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# Constructal design of a thermoelectric device

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**Abstract**

Finite time thermodynamics, a subfield of irreversible thermodynamics, is employed to model a cascaded thermoelectric generator, which incorporates all the essential features of a real heat engine. Control volume formulation of a cascaded thermoelectric element was carried out over a small but finite temperature gap to comply with the principles of irreversible thermodynamics. Three important dimensionless parameters are identified to designate poor Thomson effect; low thermal conductivity and low electrical resistivity of a good semiconductor or semimetal. For ideal values of these parameters it has been demonstrated that exactly half of the Joulean heat *arrives* at both hot and cold junction when temperature maximum passes through the longitudinal center of the thermoelectric element. In general when Thomson heat cannot be neglected, the maximum and the minimum permissible length of any individual module of a cascaded thermoelectric device is predicted involving thermoelectric properties of the materials and passing current. It is observed that maximum permissible length is devoid of dependence on applied temperature gradient whereas the minimum allowable length is not. When half the Joulean heat *affects* hot end and half the cold side, thermal conductance inventory is allocated equally between the high and low temperature side for best possible device performance. Finally, it has been argued that the choice of constancy of total conductance is not only a natural constraint but also a purely realistic design criterion for heat engines or refrigerators. Such design prescriptions those lead to the prediction of shape and structure in any flowing system is reported in open literature as constructal principle.


*Keywords:* Conductance allocation; Constructal; Endoreversible; Equipartition; Finite-time thermodynamics; Synthetic constraint; Thermoelectric device

## 1. Introduction

Thermoelectric generator is a useful and environment friendly device for direct energy conversion. Especially the capacity of Peltier and Seebeck effect to dispense with the moving parts in the realm of energy transformation from heat to electricity and vice versa is more appealing in such devices. With the advent of semiconductor materials the efficiency of a thermoelectric generator can even be an alternative for the conventional heat engines [1]. Another perspective of thermodynamic modeling of a thermoelectric generator is that it includes all the crucial features of a real heat engine in a relatively simple way where closed form expressions are obtained for the power versus efficiency characteristics [2]. Here each generic source of irreversibility is identified and quantified in this process to draw a one is to one correspondence between the conventional heat engine and the thermoelectric generator. So, the mathematical modeling of a simple thermoelectric generator can also replace the elaborate task of simulating an actual complex power plant, heat engine or refrigerator.

Much effort has been bestowed in finite time thermodynamics (FTT) [3–8], to model real heat engines. FTT modeling of thermoelectric generator presents a full-featured analysis of real engines. Since all heat engine models aim at providing a realistic margin for an improvement of actual systems, an analysis based on FTT figure-of-merit hints a more practical assessment of a maximum attainable improvement in comparison to the margin based on Carnot efficiency. Thus, the FTT modeling is a worthy endeavor. However, it is to be noted that FTT modeling does not

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## Nomenclature

| | |
|---|---|
| $A$ | surface area of the heat exchanger |
| $C$ | some finite constant, Eq. (41a) or (41b) |
| $C_A$ | some finite constant, Eq. (41d) |
| $C_B$ | some finite constant, Eq. (41f) |
| $C_K$ | some finite constant, Eq. (41c) |
| $C_Z$ | some finite constant, Eq. (41e) |
| $F_H$ | fraction of Joulean heat affecting high temperature heat source, Eq. (18) |
| $F_L$ | fraction of Joulean heat affecting low temperature sink, Eq. (18) |
| $I$ | total current |
| $J_x$ | electrical current density vector along $x$-direction |
| $K$ | thermal conductance, Eq. (25b) |
| $K_H$ | thermal conductance of high temperature side heat exchanger |
| $K_L$ | thermal conductance of low temperature side heat exchanger |
| $L$ | length of the leg of a thermoelectric device |
| n | a type of semiconductor material |
| p | type of semiconductor material |
| $\dot{Q}_H$ | heat transfer rate from the high temperature source |
| $\dot{Q}_H^*$ | heat flow rate to the hot end, Eq. (12) |
| $\dot{Q}_{*H}$ | heat flow rate to the hot end, Eq. (23) |
| $\dot{Q}_J$ | Joulean heat transport rate, Eq. (14) |
| $\dot{Q}_k$ | conducted heat transport rate, Eq. (13) |
| $\dot{Q}_L$ | heat transfer rate from the low temperature sink |
| $R$ | total electrical resistance, Eq. (25a) |
| $T$ | temperature distribution function, Eq. (1) |
| $T_{LC}$ | low temperature level at which the device actually receives the heat |
| $T_{HC}$ | high temperature level at which the device actually receives the heat |
| $\Delta T$ | applied temperature gap across a module |
| $U$ | overall heat transfer coefficient |
| $x$ | coordinate direction |
| $z$ | figure of merit, Eq. (39a) |

*Greek symbols*

| | |
|---|---|
| $\alpha$ | Seebeck coefficient of the material |
| $\varepsilon$ | effectiveness of heat exchanging equipment |
| $\gamma$ | cost of unit conductance |
| $\kappa$ | thermal conductivity of the material |
| $\kappa_e$ | electrical conductivity |
| $\kappa_l$ | lattice thermal conductivity |
| $\lambda$ | a dimensionless parameter, Eq. (6b) or (15) |
| $\Lambda$ | a dimensionless parameter, Eq. (6a) |
| $\theta$ | nondimensional temperature, Eq. (4a) |
| $\theta_*$ | nondimensional temperature distribution without Thomson heat consideration, Eq. (19) |
| $\theta^*$ | nondimensional temperature distribution with Thomson heat consideration, Eq. (8) |
| $\rho$ | electrical resistivity of the material |
| $\sigma$ | electrical conductivity or reciprocal of electrical resistivity of the material |
| $\tau$ | Thomson coefficient of the material |
| $\xi$ | dimensionless length of the device arm, Eq. (4b) |
| $\xi^*$ | location of maximum temperature with Thomson heat consideration, Eqs. (9), (11a) and (11b) |
| $\xi_*$ | location of maximum temperature without Thomson heat consideration, Eqs. (21) and (22) |
| $\zeta$ | dimensionless temperature, Eq. (20) |

*Subscripts*

| | |
|---|---|
| e | quantities of electrical origin |
| l | quantities of lattice thermal origin |
| H | quantities related to high temperature source |
| J | quantities related to component of Joulean heat |
| L | quantities related to low temperature sink |
| max | maximum |
| min | minimum |
| n | n-type material |
| p | p-type material |
| $x$ | along $x$ coordinate direction |
| * | quantities pertaining to no Thomson heat consideration |

*Superscripts*

| | |
|---|---|
| * | quantities pertaining to Thomson heat consideration |
| 0 | constancy of physical parameter |

*Symbols*

| | |
|---|---|
| $\langle\ \rangle$ | represents an averaged quantity |
| $\Delta$ | represents a difference |

stipulate the highest ceiling for the efficiency rather only dictates the lower bound of the optimal efficiency of a heat engine performance affected by finite heat transfer rate irreversibility [9]. In practice, heat engines can operate between the two extreme limits: one is the reversible or maximum efficiency operation and another is the irreversible or maximum power condition. However, in practical situation the optimum design criterion is a compromise between the efficiency and power output. In the terminologies of thermoeconomics [10], optimum operating point is a trade off between the cost of fuel and the cost of installed hardware.

In a FTT model generally all possible irreversibilities are attributed only to the heat transport process external to the engine and not to the internal conversion of heat into power [11]. For a FTT model of a thermoelectric generator internal irreversibilities remain in series. Bypass heat leak

[12] incorporated into the modeling is an additional shunt among other possible alternatives [13–15] that make the engine to operate irreversibly. In the present study, bypass heat leak is identified as a major contribution to the measure of internal irreversibility. The conducting mechanical support, which is the locus of heat transfer across a finite temperature gap, is the geometrical path of irreversibility transport. The bypass heat leak phenomenon retains all the essential features of irreversibility of the engine and offers an elegant mathematical perspective for engine modeling. Though in a thermoelectric generator Joulean heating itself remains as an inherent source of irreversibility, bypass heat leak has normally higher orders of magnitude than internal irreversibility alone in the range of optimum engine performance.

The architecture of optimized flow-system, in general, is a commonplace occurrence in engineering and nature. Solutions of many challenges have been unified under the single encompassing physics based theory, the constructal principle [16], which conceives that the geometry (shape and structure) is generated in pursuit of global performance subject to global constraints, in flow systems the geometry of which is free to vary. In this context, a thermoelectric device can be thought of a flowing system through which heat and current flows. Calculation of efficiency of thermoelectric device is reported in open literature [17]. There the conditions and consequences of heat transport between the heating and the cooling medium and the junctions are not addressed. The objective of the present investigation aims at reporting finite time irreversibility of heat transport mechanism; the distribution of Joulean heat into the hot and cold space and its consequences on optimal allocation of heat exchanger inventory and finally to predict the geometrical shape and size of each individual module of a cascaded thermoelectric device. As it is pointed out in a treatise [18] that different assumptions can lead to different results, the role of assumptions in describing the model will also be stressed.

## 2. The physical model

To manifest the effect of electric current and heat transfer irreversibilities on the thermal efficiency of a thermoelectric power generator, we consider the two-leg assembly of the basic components of a device as is shown in Fig. 1. The hot junction is maintained at a high temperature level $T_{\mathrm{HC}}$ and it receives a net heat transfer rate $\dot{Q}_{\mathrm{H}}$. Similarly, the cold end of the two-leg arrangement is held at constant temperature $T_{\mathrm{LC}}$ such that the net heat rejection rate by it is $\dot{Q}_{\mathrm{L}}$. Ideally, the two legs are one-dimensional conductors along which $x$ is directed from $T_{\mathrm{HC}}$ to $T_{\mathrm{LC}}$. The potential difference generated due to Seebeck effect causes to flow a total current $I$ through the total electrical resistance $R$ of the elementary module of length $L$ of a cascaded system.

The two legs n and p are generally chosen to be of dissimilar semiconductors or semimetals. In a conventional junction design, hot ends of the two legs are both electrically and thermally connected through a highly conductive material. Thermodynamically this arrangement is equivalent to that of simple design in which n- and p-legs are joined end to end. The lateral surfaces of both the legs are insulated electrically and thermally to prevent the contact from each other. Additionally, the cold ends of the two legs are either insulated only electrically or situated separately from each other.

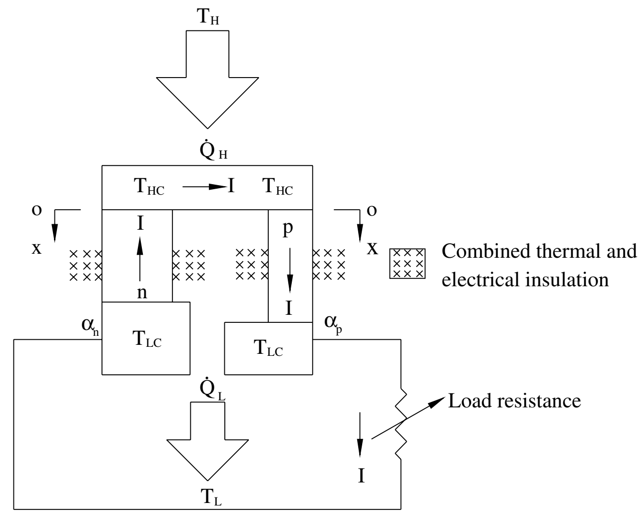

Fig. 1. Basic elements of a thermoelectric power generator.

In literature, thermodynamics of irreversible process is applied to a thermocouple where the legs may have an arbitrary shape and size; the composition may be inhomogeneous and anisotropic for the transport quantities and the properties of the materials are arbitrary functions of temperature field. Since, the maximum thermal efficiency of the device is independent of the shape of the leg of a thermoelectric element [19], in our present study its shape and size is immaterial. The geometry and physical property of the n-leg generally differs from those of p-leg. Here we will cast the problem using control volume approach along with the method of average parameters [17].

## 3. Control volume formulation of a single thermoelectric element

We will seek the temperature distribution along the device leg, as it is one of chief importance for the evaluation of thermal efficiency of the device. Under steady state condition for the divergence of the flux vector, total energy remains constant along any coordinate direction of a dimensional space. With reference to Fig. 2, specializing along $x$-direction for each leg we obtain [20]

$$TJ_x\left(\frac{\partial\alpha}{\partial x}\right)_T + \tau J_x\frac{\mathrm{d}T}{\mathrm{d}x} - \rho J_x^2 - \frac{\mathrm{d}}{\mathrm{d}x}\left(\kappa\frac{\mathrm{d}T}{\mathrm{d}x}\right) = 0 \quad (1)$$

where $J_x$ is the electrical current density vector along $x$-direction, $T$ is the temperature distribution function, $\kappa$ is the thermal conductivity of the conductor, $\rho$ is the

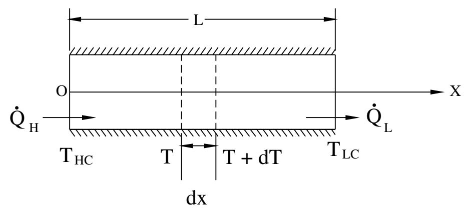


Fig. 2. A cascading thermoelectric element exposed to simultaneous current and heat flow.

electrical resistivity of the conductor, $\alpha$ and $\tau$ are the Seebeck and the Thomson coefficients, respectively.

The solution of this equation for temperature distribution demands a specification of the dependence of $\alpha$, $\kappa$, $\rho$ and $\tau$ on $x$ or $T$. One viable approximation consists of replacing all transport coefficients by their averages [17]. In this spirit, the first term in Eq. (1) drops out and we arrive at the equation

$$\langle\kappa\rangle\frac{d^2T}{dx^2} - \langle\tau\rangle J_x\frac{dT}{dx} + \langle\rho\rangle J_x^2 = 0 \tag{2}$$

where the symbol $\langle\ \rangle$ represents an averaged quantity.

Before attempting to solve the resulting simplified equation, it is to be noted that the approximation method is valid only if

$$T_{HC} \approx T_{LC} \tag{3a}$$

but

$$T_{HC} > T_{LC} \tag{3b}$$

such that the temperature difference across the thermoelectric element $\Delta T = T_{HC} - T_{LC} > 0$. These mathematical restrictions are of little practical interest, since for operation of the device at higher efficiency, temperature difference should be as high as possible on the whole of the thermoelectric device. On the contrary, for very high temperature, the phenomenological representation of irreversible process is inappropriate. Hence, the assumption of negligible temperature gap is consistent with the physical theory developed in literature [21]. In the real world of engineering design, it represents a cascaded system, where power generation takes place discretely in successive stages in series with each other and the power is extracted at each stage. With the increase of number of modules, the temperature gap across any module is reduced and the discrete power generation mimics the continuous power production from a single module.

Now, we will nondimensionalize Eq. (2) using

$$\theta = \frac{T - T_{LC}}{T_{HC} - T_{LC}} = \frac{T - T_{LC}}{\Delta T} \tag{4a}$$

and

$$\xi = \frac{x}{L}. \tag{4b}$$

The resulting equation takes the form

$$\frac{d^2\theta}{d\xi^2} - \Lambda\frac{d\theta}{d\xi} + \lambda = 0 \tag{5}$$

where

$$\Lambda = \frac{\langle\tau\rangle J_x L}{\langle\kappa\rangle} \tag{6a}$$

and

$$\lambda = \frac{\langle\rho\rangle(J_x L)^2}{\langle\kappa\rangle\Delta T}. \tag{6b}$$

The boundary conditions transform into

$$\theta = 1 \quad \text{at } \xi = 0 \tag{7a}$$

and

$$\theta = 0 \quad \text{at } \xi = 1. \tag{7b}$$

Solution of Eq. (5) subjected to boundary conditions (7a) and (7b) reads as

$$\theta^* = \frac{\lambda}{\Lambda}\xi + \left[\frac{1 + \frac{\lambda}{\Lambda}}{1 - \exp(\Lambda)}\right]\exp(\Lambda\xi) + \frac{\exp(\Lambda\xi) + \frac{\lambda}{\Lambda}}{\exp(\Lambda) - 1}. \tag{8}$$

Now we would like to locate the regime of maximum temperature. This is an important observation when we mimic a thermoelectric device with that of heat engine [2]. In a finite-time heat engine model there is a continuous variation of temperature from heat source to heat sink along the physical path of energy transport. When both the legs of the thermoelectric device is of the same length, the location of maximum temperature in either of the leg of the thermoelectric generator is obtained by setting $\frac{d\theta^*}{d\xi} = 0$, which yields

$$\xi^* = \frac{1}{\Lambda}\ln\left\{\frac{1}{1 + \frac{\Lambda}{\lambda}}\left[\frac{\exp(\Lambda) - 1}{\Lambda}\right]\right\}. \tag{9}$$

Now, we would like to prescribe some design conditions those will cause the temperature maximum to pass through the geometrical mid-point of the module of a cascaded thermoelectric device. Each individual module can be thought of an independent heat engine or one-dimensional insulation system. For a narrow temperature gap across the module, the temperature maximum passes through the mid-point of the device and it experiences a minimum entropy generation or equivalently maximum efficiency condition [22]. From the definition (6a) the imposition of the design criterion $\Lambda \to 0$ leads to the specification of maximum permissible length of any individual thermoelectric module as

$$L_{max} = \frac{\langle\kappa\rangle}{\langle\tau\rangle}\cdot\frac{1}{J_x}. \tag{10a}$$

Again the design prescription $\frac{\Lambda}{\lambda} \to 0$ stipulates from the definitions (6a) and (6b) that the minimum permissible length of the individual thermoelectric module as

$$L_{min} = \frac{\langle\tau\rangle}{\langle\rho\rangle}\cdot\frac{\Delta T}{J_x}. \tag{10b}$$

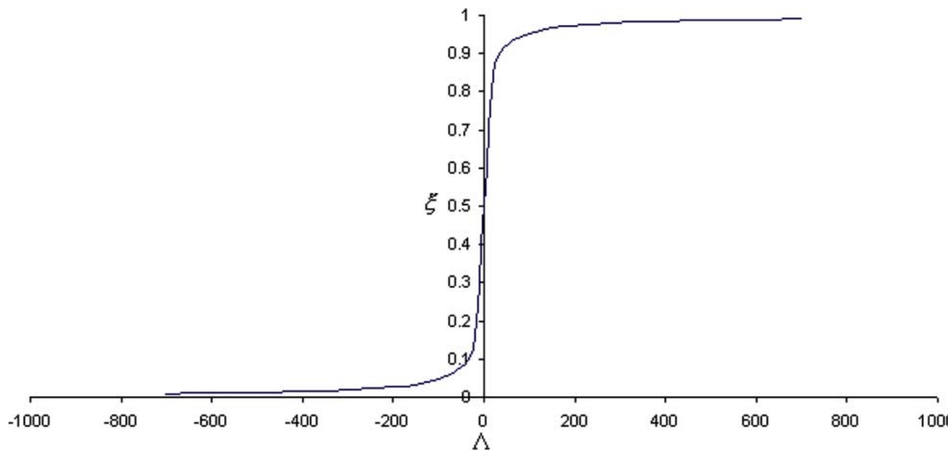


Fig. 3. Location of maximum temperature in concurrence with Thomson effect.

Thus it is to be observed that the minimum length of the device arm depends on the applied temperature gap whereas maximum permissible length is devoid of dependence of such imposed temperature gradient.

Eq. (10b) as a design criterion transforms Eq. (9) into the form

$$\xi^* = \frac{1}{\Lambda}\ln\left[\frac{\exp(\Lambda)-1}{\Lambda}\right]. \tag{11a}$$

Eq. (11a) is plotted in Fig. 3, which shows that for $\Lambda \to 0$, the relation between $\xi$ and $\Lambda$ is linear. Expanding the right side of Eq. (11a) analytically around the singular point $\Lambda = 0$ and then passing to the limit, we have

$$\lim_{\Lambda\to 0}\xi^* = \lim_{\Lambda\to 0}\frac{1}{\Lambda}\ln\left\{\frac{1}{\Lambda}\left[\left(1+\Lambda+\frac{\Lambda^2}{2!}+\frac{\Lambda^3}{3!}+\cdots\right)-1\right]\right\} = \frac{1}{2}. \tag{11b}$$

Thus Eq. (11b) clearly demonstrates that for $\Lambda = 0$, temperature maximum passes through the geometric mid point of the conductor as the electrical current changes the direction. As long as Eqs. (3a) and (3b) are valid the result obtained in Eq. (11b) is physically realistic. So, in order to construct a cascaded system the length of the first junction should be half of the total permissible length of the assembly of the thermocouples. The geometric mid point will act as a heat source for the next junction and so on.

Heat flow towards the hot end is calculated by invoking Fourier conduction law as

$$\dot{Q}_{\mathrm{H}}^* = -\kappa A\frac{\partial T}{\partial x}\bigg|_{x=0} = -\dot{Q}_k\frac{\partial\theta^*}{\partial\xi}\bigg|_{\xi=0} = \dot{Q}_{\mathrm{J}}\left[\frac{\Lambda-\exp(\Lambda)+1}{\Lambda\exp(\Lambda)-\Lambda}\right] \tag{12}$$

where the conducted heat through cross-sectional area $A$ and of length $L$ is

$$\dot{Q}_k = \frac{\kappa A\Delta T}{L} \tag{13}$$

and the Joulean heat source of cross-sectional area $A$ and length $L$ is

$$\dot{Q}_{\mathrm{J}} = \rho J_x^2 AL \tag{14}$$

such that as $\Lambda \to 0$

$$\lambda \approx \frac{\dot{Q}_{\mathrm{J}}}{\dot{Q}_k}. \tag{15}$$

We calculate the ratio $\left|\frac{\dot{Q}_{\mathrm{H}}^*}{\dot{Q}_{\mathrm{J}}}\right|$ in the limit $\Lambda \to 0$ in order to examine that what proportion of Joulean heat moves to the hot end. On calculating the limit, using L'Hospital's theorem, we have

$$\lim_{\Lambda\to 0}\left|\frac{\dot{Q}_{\mathrm{H}}^*}{\dot{Q}_{\mathrm{J}}}\right| = \lim_{\Lambda\to 0}\left|\frac{\Lambda-\exp(\Lambda)+1}{\Lambda\exp(\Lambda)-\Lambda}\right| = \frac{1}{2}. \tag{16}$$

From Eq. (16) we observe that exactly half of the Joulean heat proceeds to the hot end. The first law of thermodynamics asserts that precisely fifty percent of the Joulean heat contributes to the cold junction.

If the electric current changes its direction the Thomson heat also changes its sign. It implies that if two parallel conductors of almost the same geometrical and thermoelectrical attribute are placed in communication with a single reservoir and if the same strength of current flows in opposite directions through the two conductors, the Thomson heat so generated by one conductor agrees nearly with the Thomson heat absorbed by the other. Thus, the effect of Thomson heat is almost nullified. Hence the net heat transfer interaction with the thermal reservoir comprises of rejecting only two Joulean heating rates generated by the two conductors. From Eq. (6a) it can be noted that the Thomson effect need not be absent even for a vanishingly small value of the parameter $\Lambda$ as for any individual cascading member, the passing current and length of the element is small and heat transfer irreversibility phenomenon overwhelms Thomson effect. Another mathematical way of looking into the problem is the imposition of the restrictions that $\kappa$, $\rho$ and $\alpha$ are constants. Then both $\left(\frac{\partial\alpha}{\partial x}\right)_T$ and $\tau = T\left(\frac{\partial\alpha}{\partial T}\right)_x$ vanishes, such that Eq. (2) further reduces to

$$k_0\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + \rho_0 J_x^2 = 0 \tag{17}$$

where the constant values are indicated by the zero subscript.

At the first sight it seems that the foregoing argument waives the imposition of the restriction that the temperature difference should be small if we ignore the origin of Eq. (17). It is to be noted that the constancy of these thermophysical and electrical properties demand in turn the narrow temperature range of operation of the individual element of the device. Alternatively, Eq. (17) can be readily obtained by applying first law of thermodynamics for a control volume, where conduction of heat takes place with distributed heat source, without introducing the formalism of irreversible thermodynamics. Unlike thermionic device a cascaded thermoelectric element works under narrow temperature range and hence we can neglect the very effect of radiation and convection. The distribution of temperature along a thin conductor under the influence of high current involving radiative and conductive transfer is reported in the literature [23–26] from a different perspective. Contrary to Thomson heat consideration above, in this limiting case we have the liberty to formulate the boundary conditions as follows:

$$T = T_{HC} \quad \text{at } x = 0 \tag{18a}$$

$$T = T_{LC} \quad \text{at } x = L \tag{18b}$$

and

$$T_{HC} > T_{LC}. \tag{18c}$$

The absolute value of both $T_{HC}$ and $T_{LC}$ are but small.

Nondimensional solution of Eq. (17) subjected to the boundary conditions, Eqs. (18a)–(18c) is

$$\theta_* = (1-\xi) + \frac{\zeta}{2}(\xi - \xi^2) \tag{19}$$

where

$$\zeta = \frac{\rho_0 (J_x L)^2}{\kappa_0 \Delta T} = \frac{\dot{Q}_J}{\dot{Q}_k}. \tag{20}$$

Once again the location of maximum temperature is obtained by setting $\frac{d\theta_*}{d\xi} = 0$ of Eq. (19) and the final result is

$$\xi_* = \frac{1}{2}\left(1 - \frac{2}{\zeta}\right). \tag{21}$$

Passing to the limit $\zeta \to \infty$ in Eq. (21)

$$\lim_{\zeta\to\infty} \xi_* = \lim_{\zeta\to\infty} \frac{1}{2}\left(1 - \frac{2}{\zeta}\right) = \frac{1}{2} \tag{22}$$

we notice that $\xi_*$ asymptotically approaches to the finite value $\frac{1}{2}$. So the numerical value of $\zeta$ should be high for temperature maximum to occur at the geometrical middle of the conductor. Thus even for the idealized situation when the thermoelectric element behaves like a resistor under the influence of low current, the placement of the second junction begins at the middle as if the thermoelectric element were not cascaded.

Heat flow inward the high temperature side is given by

$$\dot{Q}_{*H} = -\kappa A \frac{\partial T}{\partial x}\bigg|_{x=0} = -\dot{Q}_k \frac{\partial \theta_*}{\partial \xi}\bigg|_{\xi=0} = \dot{Q}_J\left[\frac{1}{\zeta} - \frac{1}{2}\right]. \tag{23}$$

We evaluate the ratio $\left|\frac{\dot{Q}_{*H}}{\dot{Q}_J}\right|$ in the limit $\zeta \to \infty$ as

$$\lim_{\zeta\to\infty}\left|\frac{\dot{Q}_{*H}}{\dot{Q}_J}\right| = \lim_{\zeta\to\infty}\left|\frac{1}{\zeta} - \frac{1}{2}\right| = \frac{1}{2}. \tag{24}$$

Eq. (24) confirms that only half of the Joulean heat goes to the hot end. Energy balance states that sharply half of the Joulean heat arrives at the cold end side.

## 4. Control volume formulation for the complete thermoelectric device

In order to maintain a consistency with the standard notation of analysis prevailing in the literature we will define the relationship between electrical resistance and resistivity; thermal conductance and conductivity of the thermoelectric element introduced in the foregoing section. Electrical resistance $R$ is related to its counterpart resistivity $\rho$ through

$$R = \frac{\rho L}{A}. \tag{25a}$$

Thermal conductance $K$ is dependent on conductivity $\kappa$ as

$$K = \frac{\kappa A}{L}. \tag{25b}$$

For the control volume formulation of the integrated thermoelectric device as shown in Fig. 4, we employ Newton's law of cooling [27]. First law of thermodynamics

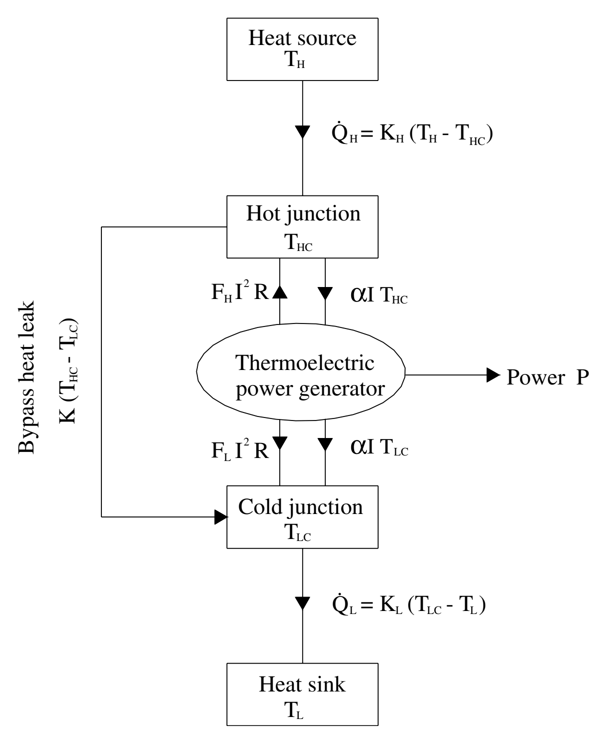


Fig. 4. Schematic diagram of a cascading finite-time thermoelectric power generator consisting of two differentially heated thermoelectric elements.

analysis neglecting Thomson effect enables us to write down the following heat transport equations in algebraic forms [2]. Finite time heat transfer rate to the hot junction $\dot{Q}_H$ is given by

$$\dot{Q}_H = K_H(T_H - T_{HC}) = \alpha I T_{HC} + K(T_{HC} - T_{LC}) - F_H I^2 R \tag{26}$$

where $K$ and $K_H$ are the thermal conductances across the reversible compartment and the hot junction, respectively. $T_H$ is the temperature of the high temperature source and $T_{HC}$ is that of thermoelectric element such that $T_{HC} \leqslant T_H$. Fraction of Joulean heat entering into the hot junction is $F_H$. Eq. (26) can be rearranged as

$$(K + K_H + \alpha I)T_{HC} - KT_{LC} - (K_H T_H + F_H I^2 R) = 0. \tag{26a}$$

Similarly, finite time heat transfer rate to the cold junction $\dot{Q}_L$ is obtained as

$$\dot{Q}_L = K_L(T_{LC} - T_L) = \alpha I T_{LC} + K(T_{HC} - T_{LC}) + F_L I^2 R \tag{27}$$

where $K_L$ is the thermal conductance across the cold junction. $T_L$ is the temperature of the low temperature sink and $T_{LC}$ is that of thermoelectric component such that $T_{LC} \geqslant T_L$. Fraction of Joulean heat entering into the cold junction is $F_L$.

Eq. (27) can be rewritten as

$$KT_{HC} - (K + K_L - \alpha I)T_{LC} + (K_L T_L + F_L I^2 R) = 0. \tag{27a}$$

For the Joulean heat distribution it is obvious that

$$F_H + F_L = 1. \tag{28}$$

Now, the system of Eqs. (26a), (27a) and (28) has four variables $T_{HC}$, $T_{LC}$, $F_L$ and $F_H$ rendering single degree of freedom. Choosing $F_H$ to be that degree of freedom we solve for $T_{HC}$ and $T_{LC}$ to obtain

$$T_{HC} = \frac{K\left[(K_H T_H + K_L T_L) + I^2 R\right] + (K_L - \alpha I)(K_H T_H + F_H I^2 R)}{K(K_H + K_L) + (K_H + \alpha I)(K_L - \alpha I)} \tag{29a}$$

and

$$T_{LC} = \frac{K\left[(K_H T_H + K_L T_L) + I^2 R\right] + (K_H + \alpha I)(K_H T_H + F_H I^2 R)}{K(K_H + K_L) + (K_H + \alpha I)(K_L - \alpha I)}. \tag{29b}$$

Now, we proceed to seek the possible set of solutions for the assumed unknown variable $F_H$ or $F_L$. Among many other methodologies [28], we devise our own based on the symmetry of the problem as follows. Eliminating $T_{HC}$ between Eqs. (26a) and (27a) and providing an expression for $T_{LC}$ from Eq. (29b) we obtain

$$\begin{aligned}&[K(K_H T_H + K_L T_L) + K_H K_L T_L] + \alpha K_L T_L I \\ &\quad + R[K + F_L K_H]I^2 + \alpha F_L R I^3 \\ &= [K(K_H T_H + K_L T_L) + K_H K_L T_L] + \alpha K_L T_L I \\ &\quad + R[F_L(K + K_H) + F_H K]I^2 + \alpha F_L R I^3.\end{aligned} \tag{30}$$

Comparing like powers of $I$, we have for the term $I^2$

$$K + F_L K_H = (K + K_H)F_L + KF_H. \tag{31}$$

Analogously, $T_{LC}$ eliminant of Eqs. (26a) and (27a) with the insertion of the expression for $T_{HC}$ from Eq. (29a) we get

$$\begin{aligned}&[K(K_H T_H + K_L T_L) + K_L K_H T_H] - \alpha K_H T_H I \\ &\quad + R[K + F_H K_L]I^2 - \alpha F_H R I^3 \\ &= [(K + K_L)K_H T_H + KK_L T_L] - \alpha K_H T_H I \\ &\quad + R[F_H(K + K_L) + F_L K]I^2 - \alpha F_H R I^3.\end{aligned} \tag{32}$$

Equating similar powers of $I$ on both sides, we have for $I^2$

$$K + F_H K_L = (K + K_L)F_H + KF_L. \tag{33}$$

Any particular solution of two identities (30) and (33) must have general functional form involving $K$, $K_H$ and $K_L$ that is

$$F_H = F_H(K, K_H, K_L) \tag{34a}$$

and

$$F_L = F_L(K, K_H, K_L). \tag{34b}$$

On following symmetry, we assume a trial solution of the form

$$\begin{aligned}F_H &= \frac{1}{2}\left(\frac{K_H K_L + KK_H + KK_H}{K_H K_L + KK_H + KK_L}\right) \\ &= \frac{1}{2}\left(\frac{K_H K_L + 2KK_H}{K_H K_L + KK_H + KK_L}\right).\end{aligned} \tag{35a}$$

Employing Eq. (35a) in Eq. (28) we obtain

$$\begin{aligned}F_L &= \frac{1}{2}\left(\frac{K_H K_L + KK_L + KK_L}{K_H K_L + KK_H + KK_L}\right) \\ &= \frac{1}{2}\left(\frac{K_H K_L + 2KK_L}{K_H K_L + KK_H + KK_L}\right).\end{aligned} \tag{35b}$$

Substituting Eqs. (35a) and (35b) into the identity Eq. (31) we have for both sides a common expression

$$C_1 = \frac{4KK_H K_L + 2K^2(K_H + K_L) + K_H^2 K_L}{2(K_H K_L + KK_H + KK_L)}. \tag{36a}$$

Similarly, inserting Eqs. (35a) and (35b) into the other identity, Eq. (33), we obtain another common expression

$$C_2 = \frac{4KK_H K_L + 2K^2(K_H + K_L) + K_H K_L^2}{2(K_H K_L + KK_H + KK_L)}. \tag{36b}$$

Eqs. (36a) and (36b) confirms that Eqs. (35a) and (35b) are a set of possible solutions for the identities (31) and (33). Further by simple inspection we observe that Eqs. (31) and (33) admit the following numerical values

$$F_H = 0 \quad \text{and} \quad F_L = 1 \tag{37a}$$

$$F_H = 1 \quad \text{and} \quad F_L = 0 \tag{37b}$$

$$F_H = \tfrac{1}{2} \quad \text{and} \quad F_L = \tfrac{1}{2}. \tag{37c}$$

Table 1
Effect of some limiting combinations of conductance allocation on Joulean heat distribution

| Range or value | $F_H$ | $F_L$ | Remarks |
|---|---|---|---|
| $K$ = Finite, $K_H$ = Finite, $K_L$ = Finite | $\frac{1}{2}$ | $\frac{1}{2}$ | $K_H = K_L$ |
| $K$ = Finite, $K_H$ = Finite, $K_L$ = Finite | 1 | 0 | $K_HK_L = -2KK_L$ |
| $K$ = Finite, $K_H$ = Finite, $K_L$ = Finite | 0 | 1 | $K_HK_L = -2KK_H$ |
| $K = 0$, $K_H = 0$, $K_L = 0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | – |
| $K$ = Finite, $K_H = 0$, $K_L$ = Finite | 0 | 1 | – |
| $K$ = Finite, $K_H$ = Finite, $K_L = 0$ | 1 | 0 | – |
| $K = 0$, $K_H$ = Finite, $K_L$ = Finite | $\frac{1}{2}$ | $\frac{1}{2}$ | – |
| $K$ = Infinite, $K_H$ = Finite, $K_L$ = Finite | $\frac{K_H}{K_H+K_L}$ | $\frac{K_L}{K_H+K_L}$ | – |
| $K$ = Finite, $K_H$ = Infinite, $K_L$ = Finite | $\frac{1}{2}\left(\frac{2K+K_L}{K+K_H}\right)$ | $\frac{1}{2}\left(\frac{K_L}{K_H+K_L}\right)$ | – |
| $K$ = Finite, $K_H$ = Finite, $K_L$ = Infinite | $\frac{1}{2}\left(\frac{K_H}{K+K_H}\right)$ | $\frac{1}{2}\left(\frac{2K+K_H}{K+K_H}\right)$ | – |
| $K$ = Infinite, $K_H$ = Infinite, $K_L$ = Infinite | 0 or 1 | 1 or 0 | – |

Now, we will examine for what combinations of $K$, $K_H$ and $K_L$ these numeric values are returned for the functional relations (31) or (33). Eqs. (35a) and (35b) along with Eq. (37a) says that

$$K_HK_L = -2KK_H. \tag{38a}$$

For Eq. (37b) to be tantamount with Eqs. (35a) and (35b) one requires that

$$K_HK_L = -2KK_L. \tag{38b}$$

Equivalency of Eqs. (35a) and (35b) with Eq. (37c) demands that

$$K_H = K_L. \tag{38c}$$

Since, $K$, $K_H$ and $K_L$ are all nonnegative quantities, only Eqs. (37c) and (38c) is physically realistic. Thus the necessary and sufficient condition for equipartition of Joulean heat produced is the equipartition of conductance allocations between high temperature and low temperature heat sources. Furthermore, when one of the three conductances runs to a very high value leaving other two in a moderate range, Joulean heat distribution again becomes unequal. Table 1 presents a compilation for the fraction of Joulean heat distribution for different limiting combinations of conductance allocations.

## 5. Consequences of equipartitioned Joulean heat

The better thermoelectric material for direct energy conversion the higher should have the value of the dimensionless group $zT$, where $T$ is the average absolute temperature and $z$ is the figure-of-merit of the thermoelectric material. This dimensionless parameter for a semimetal or semiconductor can be expressed in general as [29]

$$zT = \alpha^2 T\left(\frac{\sigma}{\kappa}\right) \tag{39a}$$

where $\sigma$ is the electrical conductivity and is the reciprocal of electrical resistivity. Thermal conductivity can further be treated as the cumulative effect of electrical conductivity $\kappa_e$ and lattice thermal conductivity $\kappa_l$ such that

$$\kappa = \kappa_e + \kappa_l. \tag{39b}$$

For higher orders of magnitude of the dimensionless group on the left side of Eq. (39a) in conjunction with Eqs. (6a) and (6b) and the conditions $\Lambda \to 0$ and $\zeta \to \infty$ we may stipulate that

$$\tau \to 0+, \quad \kappa \to 0+, \quad \rho \to 0+ \quad \text{and} \quad \Delta T \to 0+. \tag{40}$$

Thus, we conclude that the present analysis is valid even for the finite temperature difference and the effect of Thomson heat can be neglected for a good quality thermoelectric material.

Eqs. (11b) and (16) together exhibits an interesting thermodynamic property of a thermoelectric element. For temperature maximum to occur at the geometric center of a thermoelectric element, exactly half of the Joulean heat approaches to the hot end and the other half to the cold side. Similar observation is repeated through Eqs. (22) and (24).

From Eqs. (37c) and (38c) we learn that both the hot and cold junction experiences half of the net Joulean effect produced and which demands in turn equal conductance allocation on both sides. Eq. (38c) also prompts the fact that both $K_H$ and $K_L$ are finite and hence the following proposition holds

$$K_H + K_L = C \tag{41a}$$

where $C$ is some finite constant. In engineering literature conductance is denoted as a product of overall heat transfer coefficient $U$ and related surface area $A$. Thus Eq. (41a) can be rewritten as

$$U_HA_H + U_LA_L = C. \tag{41b}$$

Allocation of heat exchanger inventory was extensively investigated in connection with the optimization of refrigeration and power production both from thermodynamic [30–32] and thermoeconomic [33] viewpoint. But the final result of an optimization problem depends on the nature of imposed constraint [18]. Klein [30] considered the constraint of the type

$$\varepsilon_H U_H + \varepsilon_L U_L = C_K \tag{41c}$$

where $\varepsilon$ is the effectiveness of the heat exchanging equipment and $C_K$ is a constant. Based on the notion that both conductance and entropy generation have the same dimension, Ait-Ali [31] conceived a condition of the from

$$\frac{\dot{Q}_{\mathrm{H}}}{T_{\mathrm{H}}-T_{\mathrm{HC}}}+\frac{\dot{Q}_{\mathrm{L}}}{T_{\mathrm{LC}}-T_{\mathrm{L}}}=C_{\mathrm{A}} \tag{41d}$$

where $C_{\mathrm{A}}$ is some parametric constant. On the basis of total cost conservation of heat exchanger installation, Antar and Zubair [33] framed a relation as

$$\gamma_{\mathrm{H}}U_{\mathrm{H}}A_{\mathrm{H}}+\gamma_{\mathrm{L}}U_{\mathrm{L}}A_{\mathrm{L}}=C_{\mathrm{Z}} \tag{41e}$$

where $\gamma$ is the unit conductance cost and $C_{\mathrm{Z}}$ has a fixed value. Bejan [32] considered the maximization of power production of heat engines and refrigeration load in refrigerators with two heat reservoirs considering total area constraint for the heat exchangers on following the equation

$$A_{\mathrm{H}}+A_{\mathrm{L}}=C_{\mathrm{B}} \tag{41f}$$

where $A$ is the surface area of the heat exchanger and $C_{\mathrm{B}}$ is a constant due to some resource constraint. Treating Eq. (41b) also as a constraint, Bejan concluded that in both the cases either quantitative or qualitative equipartition of thermal conductance inventory is valid. Similar results have been echoed in other works [31–33]. Thus we propose that Eq. (41b) is the most natural constraint for such category of optimization problems. The idea of such synthetic constraint in optimization problems was advanced by the present authors [34].

## 6. Conclusions

(i) Creditably, the debatable concept of endoreversibility [35–37] in finite-time thermodynamics can be mitigated by incorporating some irreversibility factors to the reversible compartment sandwiched between two irreversible chambers. Consideration of bypass heat leak is a compensating measure to this direction. Joulean heating present in a thermoelectric generator itself is an inherent source of irreversibility and tantamount to the frictional loss in a heat engine. The discrimination between frictional heat leak and heat loss due to finite rate heat transfer was first put forward in Ref. [38]. In the current investigation bypass heat leak is identified as a major contribution to the measure of irreversibility. The sufficiency of bypass heat leak consideration in engine modeling is an established practice [2].

(ii) Our analysis shows that in a thermoelectric generator Thomson effect may be neglected in one limiting case or may not be negligible in another limiting situation even for a vanishingly small value of a certain nondimensional parameter $\Lambda$. A very high value of another dimensionless factor $\zeta$ recognizes a better figure-of-merit and the operation of the thermoelectric device as cascaded system over a small but finite temperature gap.

(iii) Three parameters $\Lambda$, $\lambda$ and $\zeta$ so identified are responsible for temperature maximum to pass through the geometrical middle of the one-dimensional physical device. This observation is in conformity with the principle of insulation design and the broader sense of engine modeling. The parallelism between the design principle of heat engine, heat exchanger and refrigerator to that of insulation system was established by a pioneering work of Bejan [22]. For the most efficient system, a stack of insulation is cooled midway. Similarly the calculation of mid point temperature is of intrinsic importance also in the application of thermionic elements where an interesting phenomenon occurs in the middle of the conductor: the temperature reaches extremum, remains constant there and Joulean heating and radiative heat transfer takes an equal share of the feeded energy [23–26].

(iv) Imposing the design prescription $\Lambda \to 0$ leads to the limit of maximum permissible length of any individual module of the cascaded thermocouple. Another design criterion $\frac{\Lambda}{\lambda} \to 0$ stipulates the minimum permissible length of such module. It is to be noticed that the maximum allowable length depends explicitly only on the thermoelectric properties of the material and the value of the passing current whereas the minimum permissible length additionally depends on applied temperature gradient.

(v) For the ideal values of these three parameters $\Lambda$, $\lambda$ and $\zeta$ it is interesting to report that exactly half of the Joulean heat *flows* into the hot end and half to the cold junction. *It is to be noted that this is not equivalent to state that half the Joulean heat affects either end* [39].

(vi) Control volume formulation for the integrated device disseminates that exactly half of the Joulean heat *affects* both the junction. This observation is duly supported by experimental evidences [40,41] in a similar class of thermoelectric devices.

(vii) It is conditioned that for finite bypass heat leak optimal conductance allocation is equipartitioned between high and low temperature side when Joulean heat *affects* both the junction equally. This instance of equipartition also conforms with the corollary [42] of constructal theory.

## References

[1] A.F. Ioffe, Semiconductor Thermoelements and Thermoelectric Cooling, Infosearch Ltd, London, 1957.
[2] J.M Gordon, Generalized power versus efficiency characteristics of heat engines: the thermoelectric generator as an instructive illustration, Am. J. Phys. 59 (1991) 551–555.
[3] F.L. Curzon, B. Ahlborn, Efficiency of a Carnot engine at maximum power output, Am. J. Phys. 43 (1975) 22–24.
[4] B. Andresen, P. Salamon, R.S. Berry, Thermodynamics in finite time, Phys. Today (1984) 62–70.
[5] A. De Vos, Reflections on the power delivered by endoreversible engines, J. Phys. D: Appl. Phys. 20 (1987) 232–236.
[6] J.M. Gordon, Maximum power point characteristics of heat engines as a general thermodynamic problem, Am. J. Phys. 57 (1989) 1136–1142.
[7] G. De Mey, A. De Vos, On the optimum efficiency of endoreversible thermodynamic process, J. Phys. D: Appl. Phys. 27 (1994) 736–739.
[8] J. Chen, The maximum power output and maximum efficiency of an irreversible Carnot heat engine, J. Phys. D: Appl. Phys. 27 (1994) 1144–1149.

[9] Z. Yan, L. Chen, The fundamental optimal relation and the bounds of power output and efficiency for an endoreversible Carnot engine, J. Phys. A: Math. Gen. 28 (1995) 6167–6175.
[10] S. Sieniutycz, P. Salamon (Eds.), Finite-Time Thermodynamics and Thermoeconomics, Taylor & Francis, New York, 1991.
[11] M.H. Rubin, Optimal configuration of a class of irreversible heat engines, Part I, Phys. Rev. A. 19 (1977) 1272–1276.
[12] A. Bejan, H.M. Paynter, Solved Problems in Thermodynamics, Dept. Mech. Eng., MIT Press, Cambridge, MA, 1976, Problem VII-D.
[13] I.I. Novikov, The efficiency of atomic power stations, J. Nucl. Energy II 7 (1957) 125–128.
[14] M.M. El-Wakil, Nuclear Power Engineering, McGraw Hill, New York, 1962, pp. 162–165.
[15] P.C. Lu, On optimal disposal of waste heat, Energy 5 (1980) 993–998.
[16] A. Bejan, Shape and Structure from, Engineering to Nature, Cambridge University Press, Cambridge, UK, 2000.
[17] B. Sherman, R.R. Heikes, R.W. Ure Jr., Calculation of efficiency of thermoelectric device, J. Appl. Phys. 31 (1960) 1–16.
[18] A. De Vos, B. Desoete, Equipartition principle in finite-time thermodynamics, J. Non-Equilib. Thermodyn. 25 (2000) 1–13.
[19] A.H. Boerdijk, Contribution to a general theory of thermocouples, J. Appl. Phys. 30 (1959) 1080–1083.
[20] T.C. Harman, J.M. Honig, Thermoelectric and Thermomagnetic Effects and Applications, McGraw Hill, New York, 1967, p. 276.
[21] S.R. De Groot, Thermodynamics of Irreversible Process, Interscience Pub. Inc., New York, 1952, pp. 141–162.
[22] A. Bejan, Entropy Generation through Heat and Fluid Flow, John Wiley & Sons, New York, 1982, pp. 173–177.
[23] S.C. Jain, Sir K.S. Krishnan, The distribution of temperature along a thin rod electrically heated in vacuo, Part I: Theoretical, Proc. Roy. Soc. Lond. A 222 (1954) 167–180.
[24] S.C. Jain, Sir K.S. Krishnan, The distribution of temperature along a thin rod electrically heated in vacuo, Part II: Theoretical, Proc. Roy. Soc. Lond. A 225 (1954) 1–7.
[25] S.C. Jain, Sir K.S. Krishnan, The distribution of temperature along a thin rod electrically heated in vacuo, Part III: Experimental, Proc. Roy. Soc. A 225 (1954) 7–18.
[26] S.C. Jain, Sir K.S. Krishnan, The distribution of temperature along a thin rod electrically heated in vacuo, Part IV: Many useful formulae verified, Proc. Roy. Soc. A 225 (1954) 19–32.
[27] P. Salamon, A. Nitzan, Finite time-optimizations of a Newton's law Carnot cycle, J. Chem. Phys. 74 (1981) 3546–3560.
[28] K. Rektorys (Ed.), Survey of Applicable Mathematics, Iliffe Books Ltd, London, 1969, pp. 70–75.
[29] G. Min, D.M. Rowe, Thermoelectric figure-of-merit barrier at minimum lattice thermal conductivity, Appl. Phys. Lett. 77 (2000) 860–862.
[30] S.A. Klein, Design consideration for refrigeration cycles, Int. J. Refrig. 15 (1992) 181–185.
[31] M. Ait-Ali, Maximum power and thermal efficiency of an irreversible power cycle, J. Appl. Phys. 78 (1995) 4313–4318.
[32] A. Bejan, Theory of heat transfer-irreversible power plants, Part I: The optimal allocation of heat exchange equipment, Int. J. Heat Mass Transfer 38 (1995) 433–444.
[33] M.A. Antar, S.M. Zubair, Thermoeconomic considerations in the optimum allocation of heat exchanger inventory for a power plant, Energy Convers. Manage. 42 (2001) 1169–1179.
[34] A.K. Pramanick, P.K. Das, Heuristics as an alternative to variational calculus for optimization of a class of thermal insulation systems, Int. J. Heat Mass Transfer (2005) 1851–1857.
[35] D.P. Sekulic, A fallacious argument in the finite-time thermodynamics concept of endoreversibility, J. Appl. Phys. 83 (1998) 4561–4565.
[36] B. Andresen, Comment on "A fallacious argument in the finite-time thermodynamic concept of endoreversibility", J. Appl. Phys. 83 (2001) 6557–6559.
[37] D.P. Sekulic, Response to "Comment on 'A Fallacious argument in the finite-time thermodynamics concept of endoreversibility'", J. Appl. Phys. 90 (2001) 6560–6561.
[38] B. Andresen, P. Salamon, R.S. Berry, Thermodynamics in finite-time: extremals for imperfect heat engines, J. Chem. Phys. 66 (1977) 1571–1577.
[39] W.H. Luke, Reply to experiment in thermoelectricity, Am. J. Phys. 28 (1960) 563.
[40] J.H. Noon, B.J. O'Brien, Sophomore experiment in thermoelectricity, Am. J. Phys. 26 (1958) 373–375.
[41] J.K. Logon, J.R. Clement, H.R. Jeffers, Resistance minimum of magnesium: heat capacity between 3 °K and 13 °K, Phy. Rev. 105 (1957) 1435–1437.
[42] A.K. Pramanick, P.K. Das, Note on constructal theory of organization in nature, Int. J. Heat Mass Transfer 48 (2005) 1974–1981.